



Generalized lax epimorphisms in the additive case[☆]

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ABSTRACT

In this paper we call generalized lax epimorphism a functor defined on a ring with several objects, with values in an abelian AB5 category, for which the associated restriction functor is fully faithful. We characterize such a functor with the help of a conditioned right cancellation of another functor, constructed in a canonical way from the initial one. As consequences we deduce a characterization of functors inducing an abelian localization and also a necessary and sufficient condition for a morphism of rings with several objects to induce an equivalence at the level of two localizations of the respective module categories.

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0. Introduction

All categories which we deal with are preadditive, i.e. there exists an abelian group structure on the hom sets, such that the composition of the morphisms is bilinear. For a category \mathcal{C} we denote by $\mathcal{C}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{Ab}$ the bifunctor assigning to every pair of objects the abelian group of all maps between them. All functors between preadditive categories are additive i.e. preserve the addition of maps. Consider a small preadditive category \mathcal{U} . Recall that a preadditive category with exactly one object is nothing but an ordinary ring with identity, therefore small preadditive categories are also called *rings with several objects*. As in the case of ordinary rings, a (*right*) *module* over \mathcal{U} (or simply, a \mathcal{U} -*module*) is a functor $\mathcal{U}^{\text{op}} \rightarrow \mathcal{Ab}$. All \mathcal{U} -modules together with natural transformations between them form an abelian, AB5 category denoted $\text{Mod}(\mathcal{U})$, where limits and colimits are computed pointwise. Moreover the Yoneda functor

$$\mathcal{U} \rightarrow \text{Mod}(\mathcal{U}), \quad \text{given by } U \mapsto \mathcal{U}(-, U)$$

is an embedding and its image form a set of (small, projective) generators for $\text{Mod}(\mathcal{U})$, therefore $\text{Mod}(\mathcal{U})$ is a Grothendieck category. This embedding allows us to identify an object $U \in \mathcal{U}$ with its image in $\text{Mod}(\mathcal{U})$, that is with the functor $\mathcal{U}(-, U)$. In what follows we freely use this identification. We denote by $\text{Hom}_{\mathcal{U}}(X, Y)$ the set of all \mathcal{U} -linear maps (i.e. natural transformations) between the \mathcal{U} -modules X and Y ; that is $\text{Hom}_{\mathcal{U}}(X, Y) = \text{Mod}(\mathcal{U})(X, Y)$.

Following [1], a functor between small non-additive categories $T : \mathcal{U} \rightarrow \mathcal{V}$ is called a *lax epimorphism*, provided that the functor

$$T_* : [\mathcal{V}^{\text{op}}, \mathcal{Set}] \rightarrow [\mathcal{U}^{\text{op}}, \mathcal{Set}], \quad T_*X = X \circ T$$

is fully faithful (Here $[\mathcal{U}^{\text{op}}, \mathcal{Set}]$ denotes the category of all contravariant functors from \mathcal{U} to the category of sets.). We shall use the same terminology in the additive case (consequently, replacing $[\mathcal{U}^{\text{op}}, \mathcal{Set}]$ with $\text{Mod}(\mathcal{U})$). We now consider a functor $T : \mathcal{U} \rightarrow \mathcal{C}$, where \mathcal{U} is a ring with several objects and \mathcal{C} is any cocomplete, abelian category. Then there is a unique, up to a natural isomorphism, colimit preserving functor $T^* : \text{Mod}(\mathcal{U}) \rightarrow \mathcal{C}$ such that $T^*\mathcal{U}(-, U) = TU$, for all $U \in \mathcal{U}$. The functor T^* has a right adjoint, namely the functor

$$T_* : \mathcal{C} \rightarrow \text{Mod}(\mathcal{U}), \quad T_*C = \mathcal{C}(T-, C) \quad \text{for all } C \in \mathcal{C}.$$

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The functors T^* and T_* will be called the *induction*, respectively the *restriction functor* associated to T , and the adjoint pair (T^*, T_*) is said to be *induced by T* . In accord with the above terminology, we call the functor T *generalized lax epimorphism*, if the associated restriction functor T_* is fully faithful.

For an additive functor F , we denote by $\text{Ker } F$ the full subcategory of the domain of F , consisting of all objects which are annihilated by F , in contrast with \ker which denotes the categorical notion of kernel.

By an *abelian localization* we understand a pair of adjoint functors between two abelian categories, with the properties that the left adjoint is exact and the right adjoint is fully faithful.

Let \mathcal{G} be a ring with several objects. Recall that a *localizing subcategory* in $\text{Mod}(\mathcal{G})$ is a full subcategory closed under subobjects, quotients, direct sums and extensions. Obviously, $\text{Ker } F$ is a localizing subcategory, provided that F is an exact, colimit preserving functor. It is well-known, that a localizing subcategory is nothing but a hereditary torsion class, so modules belonging to such a subcategory are called sometimes *torsion modules*.

Consider a localizing subcategory \mathcal{L} in $\text{Mod}(\mathcal{G})$. We call \mathcal{L} -*torsion free* (\mathcal{L} -*closed*) an object $X \in \text{Mod}(\mathcal{G})$ satisfying $\text{Hom}_{\mathcal{G}}(L, X) = 0$ (respectively $\text{Hom}_{\mathcal{G}}(L, X) = \text{Ext}_{\mathcal{G}}^1(L, X) = 0$) for all $L \in \mathcal{L}$, where $\text{Ext}_{\mathcal{G}}^1$ denotes as usually the first derived functor of $\text{Hom}_{\mathcal{G}}$. We construct, as in [3], the quotient category $\mathcal{C} = \text{Mod}(\mathcal{G})/\mathcal{L}$ together with the canonical (exact) functor $Q : \text{Mod}(\mathcal{G}) \rightarrow \mathcal{C}$, called also the quotient functor, which has a fully faithful right adjoint $R : \mathcal{C} \rightarrow \text{Mod}(\mathcal{G})$. Clearly the pair (Q, R) is an abelian localization. Then \mathcal{C} is a Grothendieck category, $\mathcal{L} = \text{Ker } Q$ and R identifies \mathcal{C} with the full subcategory of $\text{Mod}(\mathcal{G})$ consisting of all \mathcal{L} -closed modules (also see [3]). Note also that Q sends every morphism with torsion kernel and cokernel in $\text{Mod}(\mathcal{G})$ into an isomorphism in \mathcal{C} , and is universal with this property. In particular, if $F : \text{Mod}(\mathcal{G}) \rightarrow \mathcal{A}$ is an exact functor into an abelian category, which annihilates all torsion \mathcal{G} -modules, then F factors uniquely through Q .

In this paper we characterize a functor which is a generalized lax epimorphism, with the help of a conditioned right cancellation of a functor constructed in a canonical way from the initial one; see Theorem 2.2. As consequences we deduce in Corollary 2.3 a characterization of functors inducing an abelian localization, and in Corollary 2.4 an additive version of the “Lemme de comparaison” (see [2, Theorem 4.1]), giving a necessary and sufficient condition for a morphism of rings with several objects to induce an equivalence at the level of two localizations of the respective module categories. Note that we shall call conditioned epimorphism a functor satisfying the above mentioned conditional cancellation property. First we study such functors in Section 1, the main result being Theorem 1.7. We also give applications of our characterizations for some more or less classical cases. Thus we deduce the classical results concerning of (flat) epimorphisms of unitary rings (Proposition 3.2 and Corollary 3.3), but also the main result of Krause’s paper [6], concerning epimorphisms up to direct factors (Proposition 3.5). Another characterization of a functor which induces an abelian localization as in Corollary 2.3 is the subject of [7]. Inspired by this approach we obtain in Proposition 4.1 some sufficient conditions for a functor to be a generalized lax epimorphism. In addition we discuss an example (Example 4.4), where we clarify a point which is called “obscure” in [7].

1. Generalized closed functors and conditioned epimorphisms

We fix in this section the notations as follows: \mathcal{G} is a ring with several objects, \mathcal{L} is a localizing subcategory of $\text{Mod}(\mathcal{G})$, $\mathcal{C} = \text{Mod}(\mathcal{G})/\mathcal{L}$ is the corresponding quotient category, with the quotient functor $Q : \text{Mod}(\mathcal{G}) \rightarrow \mathcal{C}$, having the right adjoint $R : \mathcal{C} \rightarrow \text{Mod}(\mathcal{G})$. We consider also a morphism of rings with several objects $S : \mathcal{U} \rightarrow \mathcal{G}$.

We call *generalized \mathcal{L} -closed* a functor $F : \mathcal{G} \rightarrow \mathcal{A}$, into a cocomplete, abelian category \mathcal{A} , provided that the induced functor $F^* : \text{Mod}(\mathcal{G}) \rightarrow \mathcal{A}$ annihilates all torsion modules (that is $\mathcal{L} \subseteq \text{Ker } F^*$) and F^* preserves exactness of sequences of the form $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ with $L \in \mathcal{L}$. About the morphism of rings with several objects S we say that it is a *\mathcal{L} -conditioned epimorphism* if the equality $F \circ S = F' \circ S$ implies $F = F'$, provided that F is generalized \mathcal{L} -closed. Note that this implication, without supplementary condition on F , means precisely that S is an epimorphism in the category of rings with several objects (see Lemma 3.1 below). In the next proposition we characterize those functors which are generalized \mathcal{L} -closed. Note first:

Remark 1.1. A module $X \in \text{Mod}(\mathcal{G})$ is \mathcal{L} -closed, in the classical sense, if and only if the functor $X^{\text{op}} : \mathcal{G} \rightarrow \mathcal{A}b^{\text{op}}$ is generalized \mathcal{L} -closed, explaining our terminology. Indeed, it is enough to observe that the induced functor is given by

$$(X^{\text{op}})^* = \text{Hom}_{\mathcal{G}}(-, X) : \text{Mod}(\mathcal{G}) \rightarrow \mathcal{A}b^{\text{op}}.$$

Proposition 1.2. The following are equivalent for a functor $F : \mathcal{G} \rightarrow \mathcal{A}$ into a cocomplete, abelian category \mathcal{A} :

- (i) The functor $F : \mathcal{G} \rightarrow \mathcal{A}$ is generalized \mathcal{L} -closed.
- (ii) F_*A is \mathcal{L} -closed for all $A \in \mathcal{A}$, or equivalently there exists $F_* : \mathcal{A} \rightarrow \mathcal{C}$ such that $F_* \cong R \circ F_*$.
- (iii) F^* factors through Q i.e. there exists $F^* : \mathcal{C} \rightarrow \mathcal{A}$ such that $F^* \cong F^* \circ Q$.

Moreover if these conditions are satisfied, then F^* is the left adjoint of F_* .

Proof. (i) \Rightarrow (ii). Let $A \in \mathcal{A}$ and $L \in \mathcal{L}$. The isomorphism

$$\text{Hom}_{\mathcal{G}}(L, F_*A) \cong \mathcal{A}(F^*L, A) = \mathcal{A}(0, A) = 0$$

shows that F_*A is \mathcal{L} -torsion free. Further we consider a short exact sequence $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$, with N projective and $L \in \mathcal{L}$. By assumption we have $F^*M \cong F^*N$, so

$$\text{Hom}_{\mathcal{G}}(M, F_*A) \cong \mathcal{A}(F^*M, A) \cong \mathcal{A}(F^*N, A) \cong \text{Hom}_{\mathcal{G}}(N, F_*A).$$

Using this together with the exact sequence of abelian groups

$$0 = \text{Hom}_{\mathcal{G}}(L, F_*A) \rightarrow \text{Hom}_{\mathcal{G}}(N, F_*A) \rightarrow \text{Hom}_{\mathcal{G}}(M, F_*A) \\ \rightarrow \text{Ext}_{\mathcal{G}}^1(L, F_*A) \rightarrow \text{Ext}_{\mathcal{G}}^1(N, F_*A) = 0,$$

we deduce $\text{Ext}_{\mathcal{G}}^1(L, F_*A) = 0$, thus F_*A is \mathcal{L} -closed. Since R is fully faithful, this property is equivalent to the factorization of F_* through R .

(ii) \Rightarrow (iii). First we shall show that $F^*L = 0$ for all $L \in \mathcal{L}$. Indeed for all $A \in \mathcal{A}$ the isomorphism

$$\mathcal{A}(F^*L, A) \cong \text{Hom}_{\mathcal{G}}(L, F_*A) = 0$$

proves our claim. Let now $\alpha : M \rightarrow N$ be a \mathcal{G} -linear map such that $Q\alpha$ is an isomorphism. In particular, the cokernel of this map belongs to \mathcal{L} , so $F^*\alpha$ is an epimorphism, since F^* is right exact. Moreover, for all $A \in \mathcal{A}$ we have the isomorphisms (in the category of abelian group homomorphisms):

$$\mathcal{A}(F^*\alpha, A) \cong \text{Hom}_{\mathcal{G}}(\alpha, F_*A) \cong \text{Hom}_{\mathcal{G}}(\alpha, (R \circ F_*)A) \cong \mathcal{C}(Q\alpha, F_*A),$$

showing that $\mathcal{A}(F^*\alpha, A)$ is bijective, therefore $F^*\alpha$ is a split monomorphism. Thus $F^*\alpha$ is an isomorphism, so F^* factors through Q .

(iii) \Rightarrow (i) is obvious.

Using the fully faithfulness of R , we have the following natural isomorphisms, for all $A \in \mathcal{A}$ and all $C \in \mathcal{C}$:

$$\mathcal{C}(C, F_*A) \cong \text{Hom}_{\mathcal{G}}(RC, (R \circ F_*)A) \cong \text{Hom}_{\mathcal{G}}(RC, F_*A) \cong \mathcal{A}((F^* \circ R)C, A),$$

showing that $F^* \circ R \cong F^* \circ Q \circ R \cong F^*$ is the left adjoint of F_* . \square

In what follows we want to characterize the conditioned epimorphisms of rings with several objects. For an easier reference we recall the following:

Lemma 1.3 ([5, Section 4]). *With the above notations, the restriction functor*

$$S_* : \text{Mod}(\mathcal{G}) \rightarrow \text{Mod}(\mathcal{U}), \quad S_*X = X \circ S \quad \text{for all } X \in \text{Mod}(\mathcal{G})$$

has not only a left adjoint, namely the induction functor, which is determined uniquely up to a natural isomorphism by

$$S^* : \text{Mod}(\mathcal{U}) \rightarrow \text{Mod}(\mathcal{G}), \quad S^*U = SU \quad (U \in \mathcal{U}) \quad \text{and} \quad S^* \text{ is colimits preserving,}$$

but also a right adjoint, namely

$${}^*S : \text{Mod}(\mathcal{U}) \rightarrow \text{Mod}(\mathcal{G}), \quad ({}^*SX)G = \text{Hom}_{\mathcal{U}}(\mathcal{G}(S-, G), X).$$

Consequently, S_ is exact and preserves limits and colimits.*

Note that the restriction and the induction functor from the preceding Lemma agree with those defined in the introduction, after the identification of a ring with several objects with its image in the module category over that ring, via the Yoneda embedding.

Lemma 1.4. *If S is surjective on objects, then the restriction functor S_* is faithful and reflects isomorphisms.*

Proof. Since S is surjective on objects, it follows that $X \circ S = 0$ implies $X = 0$ for all $X \in \text{Mod}(\mathcal{G})$, which means S_* reflects zero objects. By Lemma 1.3, the functor S_* is exact therefore it commutes with images. But such a functor (exact and reflecting zero objects) is faithful and reflects isomorphisms. \square

Lemma 1.5. *Suppose that G is an \mathcal{L} -closed module, for all $G \in \mathcal{G}$ and S is surjective on objects. Then the following are equivalent:*

- (i) $S_* \circ R$ is fully faithful.
- (ii) $Q \circ S^* \circ S_* \cong Q$ naturally.
- (iii) $(Q \circ S^* \circ S_*)G \cong QG$ naturally, for all $G \in \mathcal{G}$.

Proof. According to Lemma 1.4, S_* is faithful, hence the arrow of adjunction

$$\mu_X : (S^* \circ S_*)X \rightarrow X$$

is an epimorphism, for all $X \in \text{Mod}(\mathcal{G})$. The arrow of the adjunction between $Q \circ S^*$ and $S_* \circ R$ is given by

$$Q\mu_{RC} : (Q \circ S^*) \circ (S_* \circ R)C \rightarrow C \quad \text{for all } C \in \mathcal{C}.$$

Clearly $S_* \circ R$ is fully faithful, exactly if $Q\mu_{RC}$ is an isomorphism for all $C \in \mathcal{C}$, or equivalently μ_X has torsion kernel for all \mathcal{L} -closed $X \in \text{Mod}(\mathcal{G})$. On the other hand (ii) is equivalent to the fact $\ker \mu_X \in \mathcal{L}$ for all $X \in \text{Mod}(\mathcal{G})$, therefore (ii) \Leftrightarrow (i) follows.

(i) \Rightarrow (iii) As we have seen, $\mu_X \in \mathcal{L}$ for all \mathcal{L} -closed $X \in \text{Mod}(\mathcal{G})$. In particular $\ker \mu_G \in \mathcal{L}$ for all $G \in \mathcal{G}$. Applying the exact functor Q to the exact sequence

$$0 \rightarrow \ker \mu_G \rightarrow (S^* \circ S_*)G \xrightarrow{\mu_G} G \rightarrow 0,$$

we obtain the desired isomorphism.

(iii) \Rightarrow (ii) For an arbitrary module $X \in \text{Mod}(\mathcal{G})$, there is an exact sequence in $\text{Mod}(\mathcal{G})$

$$0 \rightarrow Y \rightarrow \bigoplus G_i \rightarrow X \rightarrow 0.$$

We apply the colimits preserving functor $S^* \circ S_*$ (see Lemma 1.3), and the Ker–Coker lemma for the obtained diagram shows that $\ker \mu_X$ is a quotient of the \mathcal{L} -torsion module $\bigoplus \ker \mu_{G_i}$, therefore it is also \mathcal{L} -torsion. \square

Lemma 1.6. Suppose that G is an \mathcal{L} -closed module, for all $G \in \mathcal{G}$ and S is surjective on objects. If $S_* \circ R$ is fully faithful, then S is an \mathcal{L} -conditioned epimorphism.

Proof. Let $F, F' : \mathcal{G} \rightarrow \mathcal{A}$ be two functors into a cocomplete, abelian category, such that F is generalized \mathcal{L} -closed, and $F \circ S = F' \circ S$. Then we obtain in turn the following natural isomorphisms: $S_* \circ F_* \cong S_* \circ F'_*$, so $Q \circ S^* \circ S_* \circ F_* \cong Q \circ S^* \circ S_* \circ F'_*$ and $Q \circ F_* \cong Q \circ F'_*$ by Lemma 1.5; further $R \circ Q \circ F_* \cong R \circ Q \circ F'_*$, so $F_* \cong R \circ Q \circ F'_*$, since F is generalized \mathcal{L} -closed, equivalently F_* factors through R by Proposition 1.2. From the arrow of adjunction $\mathbf{1}_{\text{Mod}(\mathcal{G})} \rightarrow R \circ Q$, we obtain a natural morphism

$$F'_* A \rightarrow (R \circ Q \circ F'_*) A \cong F_* A \quad \text{for all } A \in \mathcal{A},$$

which induces the isomorphism $(S_* \circ F'_*) A \xrightarrow{\cong} (S_* \circ F_*) A$. Because S_* reflects isomorphisms, we deduce that the functors F_* and F'_* are naturally isomorphic. Therefore $F \cong F'$ naturally. But F and F' coincide on objects, S being surjective on objects. Thus $F = F'$. \square

Theorem 1.7. If $S : \mathcal{U} \rightarrow \mathcal{G}$ is bijective on objects and G is an \mathcal{L} -closed module, for all $G \in \mathcal{G}$, then the following statements are equivalent:

- (i) $S_* \circ R$ is full.
- (ii) $S_* \circ R$ is fully faithful.
- (iii) S is an \mathcal{L} -conditioned epimorphism.

Proof. (i) \Leftrightarrow (ii) is immediate, since both S_* and R are known to be faithful and (ii) \Rightarrow (iii) follows by Lemma 1.6.

For the implication (iii) \Rightarrow (i), we have to show that the abelian group homomorphism

$$\text{Hom}_{\mathcal{G}}(X, Y) \rightarrow \text{Hom}_{\mathcal{U}}(S_* X, S_* Y)$$

induced by S_* is surjective for all \mathcal{L} -closed $X, Y \in \text{Mod}(\mathcal{G})$. In order to do this, we use the argument of [6, Lemma 5], observing in addition that the functor $F : \mathcal{G} \rightarrow \mathcal{A}^{b\text{op}}$, given by $F = X^{\text{op}} \oplus Y^{\text{op}}$ is generalized \mathcal{L} -closed. \square

2. When the restriction functor is fully faithful

Let $T : \mathcal{U} \rightarrow \mathcal{C}$ be any (additive) functor, where \mathcal{U} and \mathcal{C} are two arbitrary (preadditive) categories. Following [6], the functor T has a canonical factorization $T = I \circ S$, where $S : \mathcal{U} \rightarrow \mathcal{G}$ is bijective on objects and $I : \mathcal{G} \rightarrow \mathcal{C}$ is fully faithful. Moreover, this factorization is unique up to an isomorphism of categories. Actually the objects of \mathcal{G} are the same as the objects of \mathcal{U} and $\mathcal{G}(U', U) = \mathcal{C}(TU', TU)$, for all $U', U \in \mathcal{U}$. The functor S is the identity on objects and $Su = Tu$ for all maps $u : U' \rightarrow U$ in \mathcal{U} . The functor I is the identity on maps and $IU = TU$, for all $U \in \mathcal{U}$ (see [6, Lemma 1]). Observe that, if $T(\mathcal{U})$ is the full subcategory of \mathcal{C} consisting of those objects of the form $T(U)$ with $U \in \mathcal{U}$, then the categories \mathcal{G} and $T(\mathcal{U})$ are equivalent. Indeed, if $\mathcal{U} \xrightarrow{T'} T(\mathcal{U}) \xrightarrow{T''} \mathcal{C}$ is the factorization of T through its image, then $\mathcal{U} \xrightarrow{S} \mathcal{G} \xrightarrow{I''} T(\mathcal{U})$ is the canonical factorization of T' , where I'' is the identity on maps and $I''U = T'U$ for all $U \in \mathcal{U}$. By construction T' is surjective on objects, so we deduce that I'' is an equivalence.

Assume now that the category \mathcal{C} is abelian, AB5. The canonical factorization $\mathcal{U} \xrightarrow{S} \mathcal{G} \xrightarrow{I} \mathcal{C}$ of T induces a diagram of categories and functors

$$\begin{array}{ccc} \text{Mod}(\mathcal{U}) & \xrightleftharpoons[T_*]{T^*} & \mathcal{C} \\ & \searrow S_* \quad \swarrow I_* & \\ & \text{Mod}(\mathcal{G}) & \end{array}$$

in which we have obviously $T^* \cong I^* \circ S^*$ and $T_* \cong S_* \circ I_*$ naturally.

In this section we consider a functor $T : \mathcal{U} \rightarrow \mathcal{C}$ defined on a ring with several objects with values in an abelian, AB5 category \mathcal{C} , together with its canonical factorization $\mathcal{U} \xrightarrow{S} \mathcal{G} \xrightarrow{I} \mathcal{C}$. Consider also the adjoint pair (T^*, T_*) induced by T .

Lemma 2.1. With the above notations the following are equivalent:

- (i) The functor T_* is faithful.
- (ii) The functor I identifies \mathcal{G} with a generating, small subcategory of \mathcal{C} .
- (iii) $T(\mathcal{U})$ is a generating subcategory of \mathcal{C} .

Moreover if one, hence all, of these conditions holds, then the category \mathcal{C} is Grothendieck, and the adjoint pair (I^*, I_*) is a localization. Consequently $\text{Ker } I^*$ is a localizing subcategory of $\text{Mod}(\mathcal{G})$.

Proof. (i) \Rightarrow (ii). The functor I is fully faithful by construction, so it identifies \mathcal{G} with a (small) full subcategory of \mathcal{C} . Let γ be a map in \mathcal{C} such that $I_*\gamma = 0$. Then $T_*\gamma = (S_* \circ I_*)\gamma = 0$, so $\gamma = 0$ since T_* is faithful. It follows that I_* is faithful, meaning precisely that \mathcal{G} is a small generating subcategory of \mathcal{C} . Therefore \mathcal{C} is Grothendieck, and (I^*, I_*) is a localization by Gabriel–Popescu theorem.

(ii) \Rightarrow (i). The condition (ii) is equivalent to the fact that I_* is faithful. But S_* is also faithful, by Lemma 1.4, so the same is true for $T_* \cong S_* \circ I_*$.

The equivalence (ii) \Leftrightarrow (iii) follows by the above observation, that the categories \mathcal{G} and $T(\mathcal{U})$ are equivalent. \square

Now we are in position to prove the main result of this work:

Theorem 2.2. *The functor T is a generalized lax epimorphism if and only if the following conditions hold true:*

- (1) \mathcal{G} generates \mathcal{C} ; consequently $\text{Ker } I^*$ is a localizing subcategory of $\text{Mod}(\mathcal{G})$, and \mathcal{C} is a Grothendieck category.
- (2) S is a $\text{Ker } I^*$ -conditioned epimorphism.

Proof. Provided that \mathcal{G} generates \mathcal{C} (therefore $\text{Ker } I^*$ is a localizing subcategory of $\text{Mod}(\mathcal{G})$), we shall show that every $G \in \mathcal{G}$ is a $\text{Ker } I^*$ -closed \mathcal{G} -module, in order to verify the hypotheses of Theorem 1.7. But I is fully faithful by construction, thus we have the isomorphisms:

$$\mathcal{G}(-, G) \cong \mathcal{C}(I-, IG) = I_*(IG) = (I_* \circ I^*)G,$$

for every $G \in \mathcal{G}$, proving our claim. Now we have only to combine Theorem 1.7 and Lemma 2.1. \square

Provided that \mathcal{C} is a Grothendieck category, we say that $T : \mathcal{U} \rightarrow \mathcal{C}$ satisfies *Ulmer's criterion of flatness*, if for every (finite) set of morphisms $u_i : U_i \rightarrow U$ in \mathcal{U} , with $1 \leq i \leq n$, there are objects $V_j \in \mathcal{U}$, with $j \in J$, and morphisms $u_{ij} : V_j \rightarrow U_i$, with $i \in \{1, \dots, n\}$ and $j \in J$, such that for each j we have $\sum_{i=1}^n u_i u_{ij} = 0$ and the sequence

$$\bigoplus_{j \in J} TV_j \xrightarrow{(Tu_{ij})} \bigoplus_{i=1}^n TU_i \xrightarrow{(Tu_i)} TU$$

is exact. By [11, Theorem] we have that the induced functor

$$T^* : \text{Mod}(\mathcal{U}) \rightarrow \mathcal{C}$$

is exact if and only if T satisfies Ulmer's criterion of flatness.

For a morphism $u : V \rightarrow U$ in \mathcal{U} and a submodule $X \leq U$ we denote by $(X : u) \leq V$ the inverse image of X through u . Recall from [4] that a (right) *Gabriel filter* on a ring with several objects \mathcal{U} is a family $\mathfrak{F} = \{\mathfrak{F}_U \mid U \in \mathcal{U}\}$, where each \mathfrak{F}_U is a set of subobjects of U satisfying:

GF1. $U \in \mathfrak{F}_U$ for all $U \in \mathcal{U}$.

GF2. For every morphism $u : V \rightarrow U$ in \mathcal{U} and every $X \in \mathfrak{F}_U$ it holds $(X : u) \in \mathfrak{F}_V$.

GF3. If $U \in \mathcal{U}$, then a submodule $X \leq U$ belongs to \mathfrak{F}_U , whenever there exists $Y \in \mathfrak{F}_U$ with the property $(X : u) \in \mathfrak{F}_V$ for any morphism $u : V \rightarrow U$ with $\text{im } u \leq Y$.

We know that, for every $U \in \mathcal{U}$, \mathfrak{F}_U is a filter on the lattice of submodules of U (that is $X, Y \in \mathfrak{F}_U \Rightarrow X \cap Y \in \mathfrak{F}_U$ and $X \in \mathfrak{F}_U, Y \leq U, X \leq Y \Rightarrow Y \in \mathfrak{F}_U$). Moreover there is a bijection between localizing subcategories of $\text{Mod}(\mathcal{U})$ and Gabriel filters on \mathcal{U} , given by $\mathcal{L} \mapsto \mathfrak{F}(\mathcal{L})$ for any localizing subcategory \mathcal{L} of $\text{Mod}(\mathcal{U})$, where:

$$\mathfrak{F}(\mathcal{L})_U = \{X \leq U \mid U/X \in \mathcal{L}\}, \quad \text{for all } U \in \mathcal{U}.$$

(For details concerning Gabriel filters on rings with several objects see [4, Section 2.1].)

Corollary 2.3. *The adjoint pair (T^*, T_*) induced by T is an abelian localization if and only if the following conditions hold:*

- (1) \mathcal{G} generates \mathcal{C} ; consequently \mathcal{C} is a Grothendieck category.
- (2) S is a $\text{Ker } I^*$ -conditioned epimorphism.
- (3) T satisfies Ulmer's criterion of flatness.

Moreover if these conditions are satisfied, then \mathcal{C} is the quotient of $\text{Mod}(\mathcal{U})$ modulo the localizing subcategory corresponding to the Gabriel filter \mathfrak{F} in \mathcal{U} , where

$$\mathfrak{F}_U = \{X \leq U \mid T^*X \cong TU \text{ naturally}\},$$

for all $U \in \mathcal{U}$.

Proof. The necessity and sufficiency of conditions (1), (2) and (3) in order to derive that T induces an abelian localization is an immediate consequence of Theorem 1.7 combined with Ulmer's criterion of flatness. For the last remaining statement, observe that \mathcal{C} is equivalent to $\text{Mod}(\mathcal{U}) / \text{Ker } T^*$, provided that (T^*, T_*) is a localization. But, for every submodule $X \leq U$ we have $U/X \in \text{Ker } T^*$ exactly if $T^*X \cong TU$. \square

Corollary 2.4. Let $P : \mathcal{U} \rightarrow \mathcal{U}'$ be a morphism of rings with several objects, and let \mathcal{L}' be a localizing subcategory of $\text{Mod}(\mathcal{U}')$. We consider the canonical factorization $\mathcal{U} \xrightarrow{S} \mathcal{G} \xrightarrow{I} \text{Mod}(\mathcal{U}')/\mathcal{L}'$ of the functor $T = T_{P, \mathcal{L}'} : \mathcal{U} \rightarrow \text{Mod}(\mathcal{U}')/\mathcal{L}'$ given by $TU = Q'(PU)$, for all $U \in \mathcal{U}$, where $Q' : \text{Mod}(\mathcal{U}') \rightarrow \text{Mod}(\mathcal{U}')/\mathcal{L}'$ denotes the quotient functor. Then the functor P induces an equivalence $\text{Mod}(\mathcal{U})/\mathcal{L} \rightarrow \text{Mod}(\mathcal{U}')/\mathcal{L}'$, for some localizing subcategory \mathcal{L} of $\text{Mod}(\mathcal{U})$, if and only if \mathcal{G} generates $\text{Mod}(\mathcal{U}')/\mathcal{L}'$, S is a $\text{Ker } I^*$ -conditioned epimorphism and T satisfies Ulmer's criterion of flatness. If this is the case, then we have also

$$\mathcal{L} = \{X \in \text{Mod}(\mathcal{U}) \mid P^*X \in \mathcal{L}'\}.$$

Proof. Denoting $\mathcal{L} = \text{Ker } T^*$, the functor P induces an equivalence of categories as stated if and only if T induces an abelian localization, therefore Corollary 2.3 applies. Moreover if this is the case,

$$\text{Ker } T^* = \{X \in \text{Mod}(\mathcal{U}) \mid P^*X \in \mathcal{L}'\}. \quad \square$$

Remark 2.5. Corollary 2.4 gives necessary and sufficient conditions for a morphism of two rings with several objects \mathcal{U} and \mathcal{U}' to induce an equivalence at the level of two localizations of $\text{Mod}(\mathcal{U})$ and $\text{Mod}(\mathcal{U}')$, respectively. In this sense it is an additive version of [2, Theorem 4.1] (see also [7, Corollary 4.5]). But it also gives a partial answer to a question occurring naturally in [9]: Given two Grothendieck categories, \mathcal{A} and \mathcal{B} , a pair of adjoint functors between them $R : \mathcal{A} \rightarrow \mathcal{B}$ at the right and $L : \mathcal{B} \rightarrow \mathcal{A}$ at the left, and a hereditary torsion class \mathcal{T} in \mathcal{A} , what additional hypotheses should be considered, such that $\{B \in \mathcal{B} \mid LB \in \mathcal{T}\}$ is a hereditary torsion class?

3. Ordinary epimorphisms of rings with several objects

In this section we shall see how our result generalizes the classical case of (flat) epimorphisms of rings (see [8] or [10]). Observe first that, for the localizing subcategory $\mathcal{L} = 0$ of a module category $\text{Mod}(\mathcal{G})$ over a ring with several objects \mathcal{G} , every \mathcal{G} -module is 0-closed, and every functor $F : \text{Mod}(\mathcal{G}) \rightarrow \mathcal{A}$ into a cocomplete, abelian category \mathcal{A} is generalized 0-closed. We shall say that the ring with several objects \mathcal{G}' has less objects than the ring with several objects \mathcal{G} if the cardinality of isomorphism classes of objects in \mathcal{G}' is smaller than the one of objects in \mathcal{G} .

Lemma 3.1. Consider a morphism of rings with several objects $S : \mathcal{U} \rightarrow \mathcal{G}$, which is surjective on objects. The following are equivalent:

- (i) S is a 0-conditioned epimorphism.
- (ii) S is an epimorphism in the category of rings with several objects.
- (iii) For every two morphisms of rings with several objects $F, F' : \mathcal{G} \rightarrow \mathcal{G}'$, where \mathcal{G}' has less objects than \mathcal{G} , we have $F \circ S = F' \circ S$ implies $F = F'$.

Proof. (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) are obvious.

(iii) \Rightarrow (i). Let $F, F' : \mathcal{G} \rightarrow \mathcal{A}$ be two (arbitrary) functors into a cocomplete, abelian category such that $F \circ S = F' \circ S$. Since S is surjective in objects, it follows that F and F' coincide on objects. If we consider $\mathcal{G}' = F(\mathcal{G}) = F'(\mathcal{G})$ (considered as a full subcategory of \mathcal{A}), then \mathcal{G}' has less objects than \mathcal{G} . It follows $F = F'$ by applying (iii) to factorizations through image of F and F' . \square

An immediate consequence of Theorem 2.2 and Lemma 3.1 is then the following well-known characterization of epimorphisms of unitary rings:

Proposition 3.2. Let A and B be two unitary rings, and let $\varphi : A \rightarrow B$ a unitary ring homomorphism. Then φ is an epimorphism in the category of unitary rings if and only if the restriction functor

$$\varphi_* : \text{Mod}(B) \rightarrow \text{Mod}(A), \quad \varphi_* Y = Y$$

is fully faithful.

From Corollary 2.3 follows as well the case of flat epimorphisms of rings:

Corollary 3.3. With the notations made in Proposition 3.2, consider the adjoint pair (φ^*, φ_*) , where

$$\varphi^* : \text{Mod}(A) \rightarrow \text{Mod}(B), \quad \varphi^* X = X \otimes_A B$$

is the induction functor and φ_* is the restriction functor defined above. Then this adjoint pair is a localization if and only if φ is a flat epimorphism of rings (i.e. is an epimorphism of unitary rings making B into a flat A -module).

Another interesting result concerning lax epimorphisms of rings with several objects makes the object of investigations of Krause's work [6]. In order to derive this result from our Theorem 2.2, we need the following:

Lemma 3.4. Let $I : \mathcal{G} \rightarrow \mathcal{V}$ be a morphism of rings with several objects, inducing a localization (I^*, I_*) . Then I^* and I_* are mutually inverse equivalences of categories if and only if G is a $\text{Ker } I^*$ -closed \mathcal{G} -module, for all $G \in \mathcal{G}$.

Proof. The direct implication is obvious since, if I^* is an equivalence, then $\text{Ker } I^*$ is 0.

Conversely, let $X \in \text{Mod}(\mathcal{G})$ be arbitrary. We want to show that the arrow of adjunction $X \rightarrow (I_* \circ I^*)X$ is an isomorphism. In order to do this, apply the colimit preserving functor $I_* \circ I^*$ (see Lemma 1.3) to a free presentation of X . We obtain a diagram

with exact rows:

$$\begin{array}{ccccccc} \bigoplus G'_j & \longrightarrow & \bigoplus G_i & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus (I_* \circ I^*) G'_j & \longrightarrow & \bigoplus (I_* \circ I^*) G_i & \longrightarrow & (I_* \circ I^*) X & \longrightarrow & 0 \end{array}$$

The first two vertical morphisms are isomorphisms by hypothesis, therefore the same is true for the third. \square

Proposition 3.5. Let $T : \mathcal{U} \rightarrow \mathcal{V}$ be a morphism of rings with several objects, and let $\mathcal{U} \xrightarrow{S} \mathcal{G} \xrightarrow{I} \mathcal{V}$ be its canonical factorization. Then the following are equivalent:

- (i) The functor T is a lax epimorphism.
- (ii) S is an epimorphism in the category of rings with several objects, and I induces an equivalence $\text{Mod}(\mathcal{G}) \rightarrow \text{Mod}(\mathcal{V})$.
- (iii) S is an epimorphism in the category of rings with several objects, and for every object $V \in \mathcal{V}$, there exists a finite set of objects $G_i \in \mathcal{G}$ with maps $v_i : V \rightarrow IG_i \rightarrow V$ in \mathcal{V} such that $1_V = \sum_i v_i$.

Proof. (i) \Leftrightarrow (ii). We have shown in the proof of Theorem 2.2 that G is $\text{Ker } I^*$ -closed for all $G \in \mathcal{G}$. Then the equivalence follows from Theorem 2.2, Lemmas 3.1 and 3.4.

The equivalence (ii) \Leftrightarrow (iii) follows by [6, Lemma 4]. \square

Note that a morphism of rings with several objects $T : \mathcal{U} \rightarrow \mathcal{V}$ satisfying the condition (iii) (therefore all) in Proposition 3.5 above, is called an *epimorphism up to direct factors* in [6]. Thus in Proposition 3.5 we give another proof of the main result in [6] that an epimorphism up to direct factors is exactly what we call a lax epimorphism.

4. A particular case and an example

We reset the notations and assumptions made in Section 2, namely $T : \mathcal{U} \rightarrow \mathcal{C}$ is a functor defined on a ring with several objects, with values into an abelian AB5 category, and $\mathcal{U} \xrightarrow{S} \mathcal{G} \xrightarrow{I} \mathcal{C}$ is its canonical factorization. To characterize the situation in which T induces an abelian localization (T^*, T_*) , as in Corollary 2.3, is the object of investigation in [7]. Inspired by this, we obtain sufficient conditions for T_* to be fully faithful, as may be seen in the following:

Proposition 4.1. Consider the following conditions relative to T :

- (G) $T(\mathcal{U})$ generates \mathcal{C} ; consequently \mathcal{C} is Grothendieck.
- (F) If $\gamma : T(U) \rightarrow T(U')$ is a map in \mathcal{C} , where $U, U' \in \mathcal{U}$, then there are objects $V_j \in \mathcal{U}$ and maps $u_j : V_j \rightarrow U$ and $u'_j : V_j \rightarrow U'$, with $j \in J$, such that $\gamma Tu_j = Tu'_j$ for all $j \in J$, and the sequence

$$\bigoplus TV_j \xrightarrow{(Tu_j)} TU \rightarrow 0$$

is exact in \mathcal{C} .

Then the have:

- (a) If T induces a localization then (G) and (F) hold true.
- (b) If (G) and (F) hold then T is a generalized lax epimorphism.

Proof. Suppose that T induces a localization (T^*, T_*) . The condition (G) follows by Lemma 2.1 and Corollary 2.3. We shall derive the condition (F) by computing γ via calculus of fractions: $\gamma = T^* \alpha (T^* \sigma)^{-1}$, where $\alpha : X \rightarrow U'$, $\sigma : U \rightarrow X$ are maps in $\text{Mod}(\mathcal{U})$, with $T^* \sigma$ invertible in \mathcal{C} . Chose a presentation

$$\bigoplus V_j \rightarrow X \rightarrow 0$$

of X in $\text{Mod}(\mathcal{U})$, where j runs over an arbitrary set J . For each $j \in J$, compose the map $V_j \rightarrow X$ with α , respectively σ , to obtain maps $u_j : V_j \rightarrow U$ and $u'_j : V_j \rightarrow U'$, satisfying the property $\gamma Tu_j = Tu'_j$. The required exactness of the sequence in (F) follows by the fact that $T^* \sigma : TU \rightarrow T^* X$ is an isomorphism.

Suppose now that (G) and (F) hold. The condition (G) is equivalent to \mathcal{G} generates \mathcal{C} , by Lemma 2.1, so $\text{Ker } I^*$ is a localizing subcategory of $\text{Mod}(\mathcal{G})$. In order to apply Theorem 2.2, we want to show that S is a $\text{Ker } I^*$ -conditioned epimorphism. Let now \mathcal{A} be a cocomplete, abelian category and let $F, F' : \mathcal{G} \rightarrow \mathcal{A}$ be two functors, such that F is generalized $\text{Ker } I^*$ -closed, and $F \circ S = F' \circ S$. Then F and F' coincide on objects, since S is bijective on objects. Let $g : G \rightarrow G'$ be a map in \mathcal{G} , and let $U, U' \in \mathcal{U}$ such that $SU = G$ and $SU' = G'$. Then $Ig : TU \rightarrow TU'$ is a map in \mathcal{C} . By (F) there exists objects $V_j \in \mathcal{U}$ and maps $u_j : V_j \rightarrow U$ and $u'_j : V_j \rightarrow U'$, with $j \in J$, such that $(Ig)(Tu_j) = Tu'_j$ for all $j \in J$, and $(Tu_j)_{j \in J} : \bigoplus_{j \in J} TV_j \rightarrow TU$ is an epimorphism. Since I is fully faithful, we deduce $gSu_j = Su'_j$ for all $j \in J$, therefore

$$(Fg)v_j = (F \circ S)u'_j = (F' \circ S)u'_j = (F'g)v_j,$$

where we denoted $v_j = (F \circ S)u_j = (F' \circ S)u_j$, for all $j \in J$. But the fact that $(Tu_j)_{j \in J}$ is an epimorphism means precisely that the map $(Su_j)_{j \in J}$ has torsion cokernel. Therefore, applying the right exact functor, which annihilates all $\text{Ker } I^*$ -torsion module F^* , we deduce that $(v_j)_{j \in J}$ is an epimorphism, therefore $Fg = F'g$, so $F = F'$. \square

Remark 4.2. In [7, Theorem 1.2] the functor T inducing an abelian localization is characterized by three conditions, two of which being (G) and (F) from Proposition 4.1 above. The third condition denoted (FF) in [7] is a particular case of Ulmer's criterion of flatness.

Remark 4.3. We may also observe that in [7, Theorem 3.7] is given the Gabriel filter (called there topology) on \mathcal{U} , for which the category \mathcal{C} is equivalent to the quotient category of $\text{Mod}(\mathcal{U})$ modulo that Gabriel filter (with the terminology of [7], \mathcal{C} is the category of sheaves over \mathcal{U} respecting that topology). This filter consists of some submodules (subfunctors) of free modules $U \in \text{Mod}(\mathcal{U})$ (representable functors) which are called there “epimorphic”. According to [7, Lemma 3.4] a submodule $X \leq U$ is an epimorphic subfunctor of U if and only if $T^*X \cong TU$ naturally, thus we are lead to the same Gabriel filter as in Corollary 2.3.

Example 4.4. We recall an example from [7], in order to see how our results give a more comprehensive approach of phenomena occurring there. Let (X, \mathcal{O}_X) be ringed space. We denote by $\text{PMod}(\mathcal{O}_X)$ and $\text{SMod}(\mathcal{O}_X)$ the category of presheaves, respectively sheaves of \mathcal{O}_X -modules. The sheafification functor $\text{PMod}(\mathcal{O}_X) \rightarrow \text{SMod}(\mathcal{O}_X)$ is exact and admits a fully faithful right adjoint, so we are in the situation of a localization. Further for all open subset $A \subseteq X$, consider as in [7, Section 5] the finitely generated projective presheaf U_A associated to A , and denote by G_A the corresponding sheaf, under the sheafification functor. Then

$$\mathcal{U} = \{U_A \mid A \text{ is a open subset of } X\}$$

is a generating subcategory of $\text{PMod}(\mathcal{O}_X)$, and

$$\mathcal{G} = \{G_A \mid A \text{ is a open subset of } X\}$$

generates $\text{SMod}(\mathcal{O}_X)$. Viewing \mathcal{U} and \mathcal{G} as full subcategories of $\text{PMod}(\mathcal{O}_X)$, respectively $\text{SMod}(\mathcal{O}_X)$, we know that $\text{PMod}(\mathcal{O}_X)$ is equivalent to $\text{Mod}(\mathcal{U})$ and $\text{SMod}(\mathcal{O}_X)$ is a localization of $\text{Mod}(\mathcal{G})$. Denote by S the functor $\mathcal{U} \rightarrow \mathcal{G}$ given by $U_A \mapsto G_A$, for all open subsets $A \subseteq X$. The relation between $\text{Mod}(\mathcal{U})$ and $\text{Mod}(\mathcal{G})$ is described in [7, Section 5] as “obscure”. Corollary 2.3 clarifies this relation, by observing that

$$\mathcal{U} \xrightarrow{S} \mathcal{G} \xrightarrow{I} \text{SMod}(\mathcal{O}_X),$$

where I is the inclusion functor, is the canonical factorization of the restriction at \mathcal{U} of the sheafification functor. Thus S is a $\text{Ker } I^*$ -conditioned epimorphism.

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